MATH 8 UNIT 1

SYSTEMS OF EQUATIONS AND INTRODUCTION TO ANGLES

NAME: ____________________________________

	Unit 1 Homework Checklist	
NOTES 1A	*SYSTEMS - page 1-22 in order given followed by:*	*5*
8.1	1 - 4, 15. (Do not need to do book's triangular approach)	*5*
8.3	1, 3, 8-12, 17, 31	*5*
8.5i	1 - 8,	*5*
8.2	13-18, 26	*5*
8.5ii	9, 11, 13	*5*
8.4	1-13 odd	*5*
NOTES 1B	*Angles page 23-37 in order given with completed worksheets:*	*5*
	WS pg 25 - Locating angles and reference angles - degrees	*5*
	WS pg 26-27 - getting familiar with special angles - degrees	*5*
	WS pg 31 - Locating angles and reference angles - radians	*5*
	WS pg 32-33 - getting familiar with special angles - radians	*5*
	WS pg 34-35 - special angles - mixed	*5*
	WS pg 37 - Finding points on the unit circle for key angles	*5*
B1	1, 3, 5, 7, 9-20	*5*
10.1i	1-37 odd	*5*
	SAMPLE TEST (pg 38-43)	*10*
		90

Unit 1 part A– Matrices, Systems of Equations

8.1 Linear Systems of Equations *(note: lighter coverage than book)*

Warm up: Solve the following 2X2 Linear Systems (2 equations with 2 unknowns):

$$\begin{cases} 2x+y=-1 \\ -4x+6y=42 \end{cases} \qquad \begin{cases} x-3y=5 \\ -2x+6y=4 \end{cases} \qquad \begin{cases} 2x+y=4 \\ -6x-3y=-12 \end{cases}$$

Case: ____________________ ______________________ ___________________

Linear System in two variables: ____

Solution: ____

Methods(thus far): 1)______________________2)______________________3)______________________

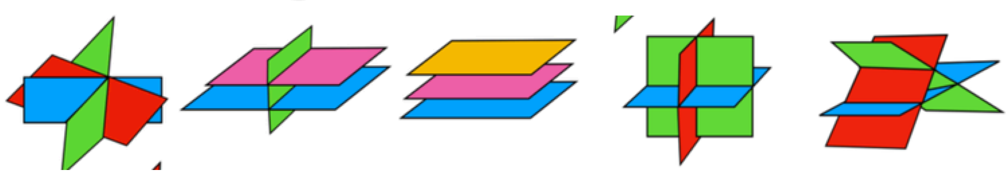
Linear Systems in 3 Variables

3 variables => 3 dimensions

Solutions are ________________________

Graph of a linear equation in 3 variables is a ____________________

Many cases for solutions to a linear system in three variables:



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_____________ ______________ ________________ ________________ _______________

Methods (thus far) 1)_________________ 2)_______________________

Example :

$$\begin{cases} 3x-2y+4z=1 \\ -4x+3y+z=-7 \\ 2x+y+3z=5 \end{cases}$$

(note, book does differently, eliminate x, upper triangular)

Special case 1 example:

$$\begin{cases} 2x+y-z=-2 \\ x+2y-z=-9 \\ x-4y+z=1 \end{cases}$$

Special case 2 example:

$$\begin{cases} x - 2y - z = 8 \\ 2x - 3y + z = 23 \\ 4x - 5y + 5z = 53 \end{cases}$$

Writing the solution to a dependent system.

8.3 Introduction to Matrices – Matrix Arithmetic

Matrix:

Size:

Square matrix:

Subscript Notation: Let a_{ij} be the entry of matrix A in row i and column j. If A is an mxn matrix, then

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \cdots\cdots & a_{1n} \\ a & a & a & a \;\cdots\cdots & a \\ \vdots & & & & \\ a & a & a & a \;\cdots\cdots & a \end{bmatrix}$$

So for matrix A= $\begin{bmatrix} 3 & 4 & 9 \\ -1 & 8 & 2 \\ 0 & -2 & 5 \end{bmatrix}$,

$a_{12} =$ ________

$a_{23} =$ ________

______ $= 0$

Matrix Operations

Definition 8.6. Matrix Addition: Given two matrices of the same size, the matrix obtained by adding the corresponding entries of the two matrices is called the **sum** of the two matrices. More specifically, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, we define

$$A + B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n}$$

If $A = \begin{bmatrix} 4 & 1 \\ -1 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 7 & -1 \\ 2 & 0 & 3 \\ -3 & 1 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 7 \\ -2 & -5 \end{bmatrix}$ find

1) A+C

2) A+B

Definition 8.7. Scalar[a] Multiplication: We define the product of a real number and a matrix to be the matrix obtained by multiplying each of its entries by said real number. More specifically, if k is a real number and $A = [a_{ij}]_{m\times n}$, we define

$$kA = k\,[a_{ij}]_{m\times n} = [ka_{ij}]_{m\times n}$$

[a]The word 'scalar' here refers to real numbers. 'Scalar multiplication' in this context means we are multiplying a matrix by a real number (a scalar). We will discuss this term momentarily.

3) Compute 3A

4) Compute 4C-A

Matrix Multiplication

Special case: Row matrix times column matrix.

Examples:

A **row vector** R is a 1 by n matrix

$$R = [r_1 \quad r_2 \quad \cdots \quad r_n]$$

A **column vector** C is an n by 1 matrix

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The **product** RC of R times C is defined as the number

$$RC = [r_1 \quad r_2 \quad \cdots \quad r_n] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = r_1c_1 + r_2c_2 + \cdots + r_nc_n$$

General matrix multiplication:

Let A denote an m by r matrix and let B denote an r by n matrix. The **product** AB is defined as the m by n matrix whose entry in row i, column j is the product of the ith row of A and the jth column of B.

Ex:

$$\begin{bmatrix} 2 & -1 & 7 \\ -3 & 0 & 4 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 6 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 4 & 1 \\ -1 & -2 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 7 \\ -2 & -5 \end{bmatrix}$ Find 1) AC

2) CA

3) A^2

Notice: Matrix multiplication is NOT ___________________________

Identity Matrix

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Find: AI_2

8.2 Gaussian Elimination and Gauss-Jordan

A way of solving linear systems of equations using matrices to help organize operations.

A system of linear equations can be represented by a matrix called an augmented matrix.

EX: System: $\begin{cases} 2x+y-z=-2 \\ x+2y-z=-9 \\ x-4y+z=1 \end{cases}$ => Augmented Matrix

EX: Augmented Matrix: $\left[\begin{array}{ccc|c} 3 & 0 & -1 & 2 \\ 7 & 9 & 2 & 1 \\ 4 & 1 & -5 & 5 \end{array}\right]$ => System

EX: Write the following Augmented matrices as a system, then solve the system:

Row Echelon Form: $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 1 & -3 \end{array}\right]$

Reduced Row Echelon Form $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{array}\right]$

Observation: If an augmented matrix is in Row Echelon Form, or Reduced Row Echelon form, it is easy to solve the corresponding system.

Definition 8.3. A matrix is said to be in **row echelon form** provided all of the following conditions hold:

1. The first nonzero entry in each row is 1.
2. The leading 1 of a given row must be to the right of the leading 1 of the row above it.
3. Any row of all zeros cannot be placed above a row with nonzero entries.

Definition 8.4. A matrix is said to be in **reduced row echelon form** provided both of the following conditions hold:

1. The matrix is in row echelon form.
2. The leading 1s are the only nonzero entry in their respective columns.

EX: Are the following in Row Echelon Form, Reduced Row Echelon Form or Neither?

$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 3 & 5 & 7 \\ 0 & 0 & 1 & -3 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 1 & 5 & 7 \\ 0 & 1 & 0 & -3 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\left[\begin{array}{ccc|c} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Gaussian/Gauss Jordan Methods

Goal of Gaussian Elimination: Given a linear system of equations, perform a series of “allowed row operations” to an augmented matrix to find a matrix in row echelon form representing an equivalent linear system. Then solve the simpler system. (If the process is continued to obtain reduced row echelon form, this is called Gauss-Jordan method.)

Illustration of the method:

Solve:

$$\begin{cases} x+2y-2z=-2 & (1) \\ -5x-9y+4z=3 & (2) \\ 3x+4y-5z=-3 & (3) \end{cases}$$

Eliminate x

$$\left.\begin{cases} 5Eq(1) \to 5x+10y-10z=-10 \\ Eq.(2) \to -5x-9y+4z=3 \end{cases}\right\} \to ADD \to \quad y-6z=-7 \quad (4)$$

$$\left.\begin{cases} -3Eq(1) \to -3x-6y+6z=6 \\ Eq.(3) \to 3x+4y-5z=-3 \end{cases}\right\} \to ADD \to \quad -2y+z=3 \quad (5)$$

Elim. Y:

$$\begin{cases} 2Eq(4) \to 2y-12z=-14 \\ Eq.(5) \to -2y+z=3 \end{cases} \quad ADD \to -11z=-11 \quad so \quad z=1$$

Augmented matrix / Row operations

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ -5 & -9 & 4 & 3 \\ 3 & 4 & -5 & -3 \end{array}\right] \cdots \begin{cases} 5R_1 + R_2 \to R_2 \\ -3R_1 + R_3 \to R_3 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & 1 & -6 & -7 \\ 0 & -2 & 1 & 3 \end{array}\right] \cdots \{ 2R_2 + R_3 \to R_3 \Rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & 1 & -6 & -7 \\ 0 & 0 & -11 & -11 \end{array}\right] \cdots \{ -\tfrac{1}{11}R_3 \to R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & 1 & -6 & -7 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Row Echelon Form

Now write the corresponding system and use back substitution to solve.

Elementary Row Operations:

Theorem 8.2. Row Operations: Given an augmented matrix for a system of linear equations, the following row operations produce an augmented matrix which corresponds to an equivalent system of linear equations.

- Interchange any two rows.
- Replace a row with a nonzero multiple of itself.[a]
- Replace a row with itself plus a nonzero multiple of another row.[b]

[a]That is, the row obtained by multiplying each entry in the row by the same nonzero number.
[b]Where we add entries in corresponding columns.

EX: Practicing Random Row Operations:

$$\left[\begin{array}{ccc|c} 3 & 0 & -1 & 2 \\ 7 & 9 & 2 & 1 \\ 4 & 1 & -5 & 5 \end{array}\right] \Rightarrow \quad -3R_2 \to R_2 \Rightarrow \left[\begin{array}{ccc|c} 3 & 0 & -1 & 2 \\ & & & \\ 4 & 1 & -5 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 0 & -1 & 2 \\ 7 & 9 & 2 & 1 \\ 4 & 1 & -5 & 5 \end{array}\right] \quad R_1 \leftrightarrow R_2 \Rightarrow \left[\begin{array}{ccc|c} & & & \\ & & & \\ 4 & 1 & -5 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 0 & -1 & 2 \\ 7 & 9 & 2 & 1 \\ 4 & 1 & -5 & 5 \end{array}\right] \Rightarrow 5R_3 + R_2 \to R_2 \Rightarrow \left[\begin{array}{ccc|c} 3 & 0 & -1 & 2 \\ & & & \\ 4 & 1 & -5 & 5 \end{array}\right]$$

The key to Gaussian elimination is to learn how to choose row operations that will yield row echelon form.

EX: Solve: $\begin{cases} 3x - y + 5z = 14 \\ x + 2y - 2z = 10 \\ x - y + 3z = 4 \end{cases}$

$$\left[\begin{array}{ccc|c} 3 & -1 & 5 & 14 \\ 1 & 2 & -2 & 10 \\ 1 & -1 & 3 & 4 \end{array}\right] \text{------------}\rightarrow$$

EX: Solve: $\begin{cases} 3x + y - z = \frac{2}{3} \\ 2x - y + z = 1 \\ 4x + 2y = \frac{8}{3} \end{cases}$

First write the augmented matrix, then obtain a 1 in position a_{11}, and then use that 1 to get zeros below it.

EX: 4X4 Gaussian Elimination / Gauss Jordan Example

Solve: $\begin{cases} x + \quad z + 2w = 6 \\ \quad y - 2z \quad = -3 \\ x + 2y - z \quad = -2 \\ 2x + y + 3z - 2w = 0 \end{cases}$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 6 \\ 0 & 1 & -2 & 0 & -3 \\ 1 & 2 & -1 & 0 & -2 \\ 2 & 1 & 3 & -2 & 0 \end{bmatrix} \xrightarrow[-2R_1+R_4\to R_4]{-R_1+R_3\to R_3} \begin{bmatrix} 1 & 0 & 1 & 2 & 6 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 2 & -2 & -2 & -8 \\ 0 & 1 & 1 & -6 & -12 \end{bmatrix} \xrightarrow[-R_2+R_4\to R_4]{-2R_2+R_3\to R_3} \begin{bmatrix} 1 & 0 & 1 & 2 & 6 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 2 & -2 & -2 \\ 0 & 0 & 3 & -6 & -9 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3\to R_3} \begin{bmatrix} 1 & 0 & 1 & 2 & 6 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & -6 & -9 \end{bmatrix} \xrightarrow{-3R_3+R_4\to R_4} \begin{bmatrix} 1 & 0 & 1 & 2 & 6 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -3 & -6 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_4\to R_4}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 6 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

This is row echelon form. If using Gaussian elimination you can stop your row operations here, write the corresponding system, and use back substitution to find the solution. If using Gauss-Jordan then continue with row operations until reduced row echelon form is achieved.

Continuing, getting zeros above the leading ones…

$$\xrightarrow[-2R_4+R_1\to R_1]{R_4+R_3\to R_3} \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[-R_3+R_1\to R_1]{2R_3+R_2\to R_2} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

From here we can see the solution, x=1, y=-1, z=1, w=2, that is (1, -1, 1, 2).

There are many other sequences of row operations that are acceptable, but they must achieve the same solution in the end. With practice, you will be able to combine more operations into each step.

Gaussian Elimination: Dependent and Inconsistent Case Examples

systems that have infinitely many solutions and systems that are inconsistent.

EXAMPLE 7 **Solving a Dependent System of Linear Equations Using Matrices**

$$\text{Solve: } \begin{cases} 6x - y - z = 4 & (1) \\ -12x + 2y + 2z = -8 & (2) \\ 5x + y - z = 3 & (3) \end{cases}$$

Solution Start with the augmented matrix of the system and proceed to obtain a 1 in row 1, column 1 with 0's below.

$$\left[\begin{array}{rrr|r} 6 & -1 & -1 & 4 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right] \underset{R_1 = -1r_3 + r_1}{\rightarrow} \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right] \underset{\substack{R_2 = 12r_1 + r_2 \\ R_3 = -5r_1 + r_3}}{\rightarrow} \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 0 & 11 & -1 & -2 \end{array}\right]$$

Obtaining a 1 in row 2, column 2 without altering column 1 can be accomplished by $R_2 = -\frac{1}{22}r_2$, by $R_3 = \frac{1}{11}r_3$ and interchanging rows 2 and 3, or by $R_2 = \frac{23}{11}r_3 + r_2$. We shall use the first of these.

$$\left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 0 & 11 & -1 & -2 \end{array}\right] \underset{R_2 = -\frac{1}{22}r_2}{\rightarrow} \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 11 & -1 & -2 \end{array}\right] \underset{R_3 = -11r_2 + r_3}{\rightarrow} \left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is in row echelon form. Because the bottom row consists entirely of 0's, the system actually consists of only two equations.

$$\begin{cases} x - 2y = 1 & (1) \\ y - \frac{1}{11}z = -\frac{2}{11} & (2) \end{cases}$$

To make it easier to write down some of the solutions, we express both x and y in terms of z.

From the second equation, $y = \frac{1}{11}z - \frac{2}{11}$. Now back-substitute this solution for y into the first equation to get

$$x = 2y + 1 = 2\left(\frac{1}{11}z - \frac{2}{11}\right) + 1 = \frac{2}{11}z + \frac{7}{11}$$

The original system is equivalent to the system

$$\begin{cases} x = \frac{2}{11}z + \frac{7}{11} & (1) \\ y = \frac{1}{11}z - \frac{2}{11} & (2) \end{cases} \quad \text{where } z \text{ can be any real number}$$

Let's look at the situation. The original system of three equations is equivalent to a system containing two equations. This means that any values of x, y, z that satisfy both

$$x = \frac{2}{11}z + \frac{7}{11} \quad \text{and} \quad y = \frac{1}{11}z - \frac{2}{11}$$

will be solutions. For example, $z = 0, x = \frac{7}{11}, y = -\frac{2}{11}$; $z = 1, x = \frac{9}{11}, y = -\frac{1}{11}$; and $z = -1, x = \frac{5}{11}, y = -\frac{3}{11}$ are some of the solutions of the original system. There are, in fact, infinitely many values of x, y, and z for which the two equations are satisfied. That is, the original system has infinitely many solutions. We will write the solution of the original system as

$$\begin{cases} x = \frac{2}{11}z + \frac{7}{11} \\ y = \frac{1}{11}z - \frac{2}{11} \end{cases} \quad \text{where } z \text{ can be any real number}$$

or, using ordered triplets, as

$$\left\{(x, y, z) \middle| x = \frac{2}{11}z + \frac{7}{11}, y = \frac{1}{11}z - \frac{2}{11}, z \text{ any real number}\right\}$$

We can also find the solution by writing the augmented matrix in reduced row echelon form. Starting with the row echelon form, we have

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{2}{11} & \frac{7}{11} \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 = 2r_2 + r_1$

The matrix on the right is in reduced row echelon form. The corresponding system of equations is

$$\begin{cases} x - \frac{2}{11}z = \frac{7}{11} & (1) \\ y - \frac{1}{11}z = -\frac{2}{11} & (2) \end{cases} \quad \text{where } z \text{ can be any real number}$$

or, equivalently,

$$\begin{cases} x = \frac{2}{11}z + \frac{7}{11} \\ y = \frac{1}{11}z - \frac{2}{11} \end{cases} \quad \text{where } z \text{ can be any real number}$$

EXAMPLE 8 **Solving an Inconsistent System of Linear Equations Using Matrices**

Solve: $\begin{cases} x + y + z = 6 \\ 2x - y - z = 3 \\ x + 2y + 2z = 0 \end{cases}$

Solution Begin with the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 6 \\ 2 & -1 & -1 & 3 \\ 1 & 2 & 2 & 0 \end{array}\right] \underset{\substack{R_2 = -2r_1 + r_2 \\ R_3 = -1r_1 + r_3}}{\rightarrow} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 6 \\ 0 & -3 & -3 & -9 \\ 0 & 1 & 1 & -6 \end{array}\right] \underset{\text{Interchange rows 2 and 3.}}{\rightarrow} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -6 \\ 0 & -3 & -3 & -9 \end{array}\right] \underset{R_3 = 3r_2 + r_3}{\rightarrow} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 0 & -27 \end{array}\right]$$

This matrix is in row echelon form. The bottom row is equivalent to the equation

$$0x + 0y + 0z = -27$$

which has no solution. The original system is inconsistent.

8.5i Determinants (not covering extensive properties as book does)

A determinant is a number corresponding to a square matrix, computed by following the processes described below. We can use determinants in a new method for solving linear systems called Cramer's Rule which we will discuss later (8.5ii). Determinants have many properties and uses. You will learn more about determinants in Math 10.

2X2 Determinant:

If A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the determinant of A, denoted det(A) or |A| is computed as follows:

det(A) = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = ______________________ Examples:

General nxn determinants.

First some terminology:

The minor , M_{ij}, of entry a_{ij} is defined to be the determinant of the matrix remaining when row i and column j is deleted from matrix A.

The cofactor , C_{ij}, of entry a_{ij} is defined to be $(-1)^{i+j}M_{ij}$ Note: this means that the cofactor is either the same as, or the opposite of the minor, depending on whether $i+j$ is even or odd.

$$A = \begin{bmatrix} 5 & 7 & -1 \\ -2 & 0 & 3 \\ -3 & 1 & 2 \end{bmatrix}$$

A helpful tool for determining whether the sign of the cofactor is the same as or opposite to the sign of the minor. (that is, whether $(-1)^{i+j}$ is positive or negative) is called the Array of Signs: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Now, to find the determinant of matrix A, we expand across any row, or down any column by taking the sum of, the product of, each entry with its cofactor.

$$\begin{vmatrix} 5 & 7 & -1 \\ -2 & 0 & 3 \\ -3 & 1 & 2 \end{vmatrix} = 5\left(\quad\right)\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} + 7\left(\quad\right)\begin{vmatrix} -2 & 3 \\ -3 & 2 \end{vmatrix} + (-1)\left(\quad\right)\begin{vmatrix} -2 & 0 \\ -3 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 5 & 7 & -1 \\ -2 & 0 & 3 \\ -3 & 1 & 2 \end{vmatrix} =$$

$$\begin{vmatrix} 5 & 7 & -1 \\ -2 & 0 & 3 \\ -3 & 1 & 2 \end{vmatrix} =$$

This method extends to any nxn matrix with the array of signs continuing in the checkerboard pattern.
Note: It is helpful to expand across a row/column with zeros.

$$\begin{vmatrix} 2 & 1 & -3 & 0 \\ -4 & -1 & 0 & 2 \\ 5 & -2 & 3 & 4 \\ 0 & 3 & 1 & 6 \end{vmatrix}$$

Ans: -494

8.5ii Cramer's Rule for solving Linear Systems (adjoints not covered)

We can generate a formula for solving a system of equations by solving the general system:

$$\begin{cases} ax+by=r \\ cx+dy=s \end{cases}$$

So if D is the determinant of the coefficient matrix: D= $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Dx is like D, but with x's column replaced by the RHS. Dx= $\begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix}$

Dy is like D, but with with y's column replaced by the RHS. Dy= $\begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix}$

Then $x=\dfrac{D_x}{D}$ and $y=\dfrac{D_y}{D}$ are the solutions to the equation (D not equal zero). Cramer's rule is particularly useful when the numbers are complicated.

Ex: $\begin{cases} 3x+\frac{1}{2}y=-4 \\ \dfrac{8}{3}x+y=2 \end{cases}$

This method extends to larger nxn linear systems.

$$\begin{cases} 2x+y-z=3 \\ -x+2y+4z=-3 \\ x-2y-3z=4 \end{cases}$$

<u>8.4 Inverse Matrices</u> (following text closely)

Much like ordinary algebraic equations, we may be asked to solve matrix equations.

Ex: If $A=\begin{bmatrix} 2 & 1 \\ 9 & 5 \end{bmatrix}$, $B=\begin{bmatrix} 3 & -1 \\ 2 & -4 \end{bmatrix}$, $X=\begin{bmatrix} x & y \\ z & w \end{bmatrix}$, solve the matrix equation 3A-2X=B for X

Ex: If $A=\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \end{bmatrix}$, solve the matrix equation AX=B.

We seek a matrix such that $A^{-1}A = AA^{-1} = I$. The matrix A^{-1}, if it exists, is called A inverse. (Note: A^{-1} does not mean $\frac{1}{A}$ here.

How do we find A^{-1}? Consider the following example, which although not how we will actually find inverses, will give us an idea why the method we will learn works.

Ex to motivate inverse process (from text): Find the inverse if A= $\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$

<u>Method for finding A^{-1} :</u>

$$[\ A \mid I_n\ ] \xrightarrow{\text{Gauss Jordan Elimination}} [\ I_n \mid A^{-1}\ ]$$

Using this method on the above matrix:

Using Matrix Equations and Matrix Inverses to solve linear systems.
Any linear system can be written in the form AX=B so if we could solve this type of equation, we can use this process as a new way of solving linear systems.

Ex: Now using the inverse above, we can finish the last example.
If $A=\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}1\\ 3\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$, solve the matrix equation AX=B.

Notice, the above matrix equation is equivalent to the system:

Any system of linear equations can be written in the form AX=B and solved in this manner.

Ex: Solve $\begin{cases}4x+6y=-5\\ 2x+3y=3\end{cases}$ by writing it as a matrix equations and solving the matrix equations

Example: Given $A=\begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix}$, find A^{-1}

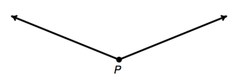
Tip: You can check your answer *as you go* since $A^{-1}A$ should equal I

EX: Solve $\begin{cases} x+y=5 \\ -x+3y+4z=7 \\ 4y+3z=4 \end{cases}$

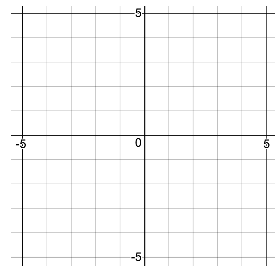
Unit 1 part 2– Introduction to Angles

B1 Angles and Degree measure (read text pg 1390+)

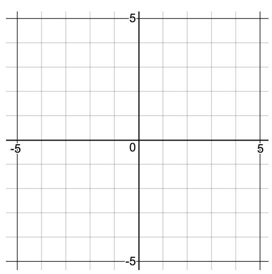
Terminology: Initial side, terminal side, standard position, positive/negative direction, complementary, supplementary, right angle, quadrantal angle, coterminal, greek names, etc.



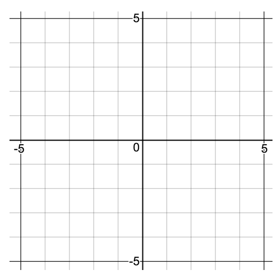
An angle with vertex *P*.



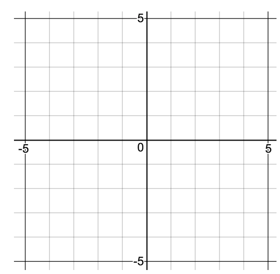
Degree Measure ______________________________



Quadrantal Angles
(1/4 of revolution)



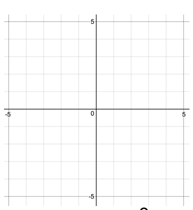
45 degrees
(1/8 revolution)
Divide quadrant in half



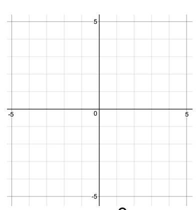
30, 60 degrees
(1/12 , 2/12 of revolution)
Divide quadrant in thirds

Locating Angles – Reference Angles

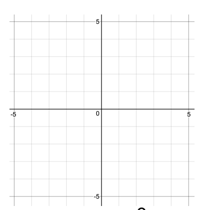
Make a rough sketch of each of the following angles in standard position. HOW did you decide where they were located?



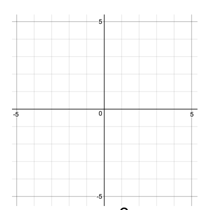
160°



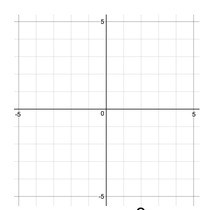
250°



400°



-178°

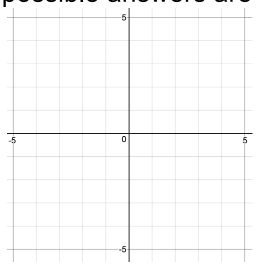


-260°

Reference angles can help us determine where an angle is located. A reference angle is the **acute** angle formed by the terminal side of a given angle, θ, and the nearest portion of the x-axis. Find the reference angles for each of the angles above.

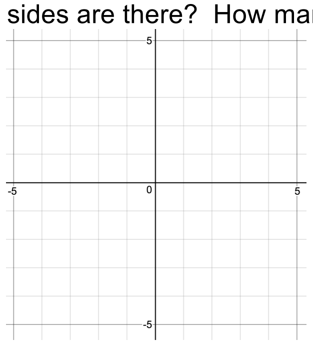
Going "backwards", sometimes we are given a reference angle and a quadrant corresponding to the terminal side of the angle and asked to locate the angle.

EX: Sketch an angle with reference angle of 10 degrees whose terminal side is in Quadrant 3. What is the measure of this angle? Give an angle coterminal with this angle. How many possible answers are there?

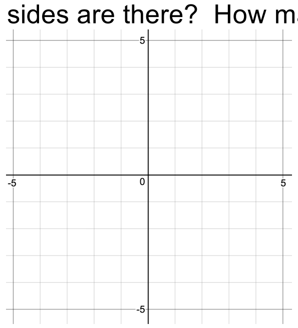


Other times the quadrant is not specified.

EX: Sketch terminal sides of all angles having reference angle of 30 degrees. How many such terminal sides are there? How many possible angles?



EX: Sketch terminal sides of all angles having reference angle of 83 degrees. How many such terminal sides are there? How many possible angles?

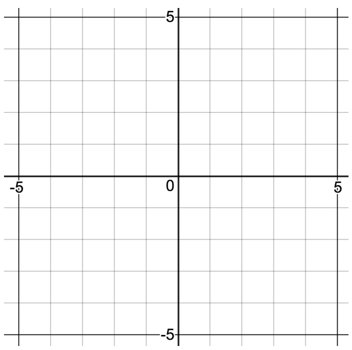


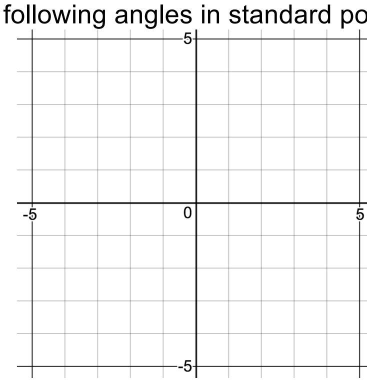
Converting Decimal Degrees $\Leftrightarrow$ Degree-Minute-Second (DMS) : Read book pg 1392-1393.

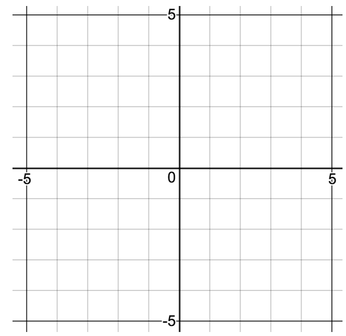
1 degree = 60 minutes , $1^\circ = 60'$
1 minute = 60 seconds, $1' = 60''$

Worksheet: Locating angles and REFERENCE ANGLES - degrees

(1) Make a rough sketch of each of the following angles in standard position and give the reference angle







170^{o} 312^{o} -102^{o}

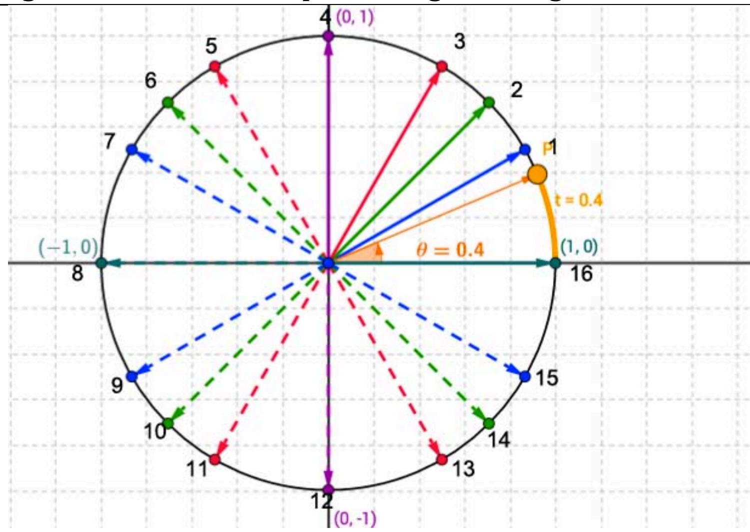
Reference Angle:

---

(2) For each of the following acute angles, find 4 angles, one in each quadrant, having the given angle as a reference angle.

	Q1	Q2	Q3	Q4
20^{o}				
87^{o}				
θ^{o} (think!)				

Worksheet:Getting Familiar with the Special Angles – Degrees



Given that all the "blue angles" have a reference angle of 30 degrees write the angle measure for each of the blue angles. (see link on math 8 page for color)

1) ___________________________ (note: the angle numbers are just for reference on this worksheet)

7)___________________________

9) ___________________________

15)___________________________

Given that all the "green angles" all have a reference angle of 45 degrees , write the angle measure for each of the green angles.

2) ___________________________

6)___________135 degrees________________

10) ___________________________

14)___________________________

Given that all the "red angles" have a reference angle of 60 degrees write the angle measures for each of the red angles.

3) ___________________________

5)___________________________

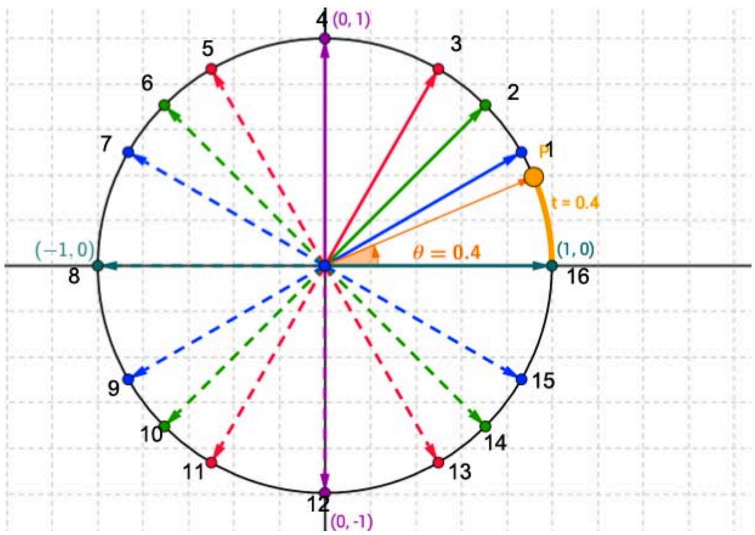
11) ___________________________

13)___________________________

(worksheet continued next page)

(worksheet contd)
Locating Special Angles - Degrees

The “blue angles” all have a reference angle of 30 degrees .
The “green angles” all have a reference angle of 45 .
The “red angles” all have a reference angle of 60 degrees .
(ignore the orange here)



Locate the following angle and write the corresponding number for each of the following angles. (You need to get quick at this)

135°________6______ 315°_______________ 180°_______________

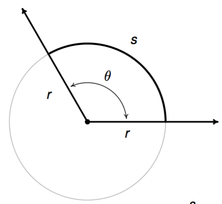
210° ______________ 120°________________ 300°________________

150° ______________ 420°________________ 765°______________

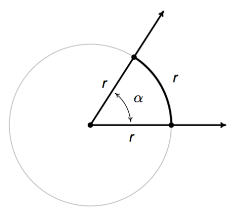
–90°______________ -210°________________ –270° ______________

10.1i Angles and Radian Measure

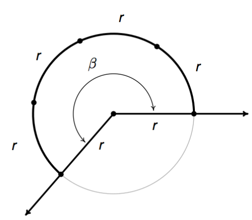
Another way of measuring angles comes historically from measuring the length, s, of an arc subtended by an angle, θ, whose vertex is at the center of the circle of radius r. (Central Angle).



We know the circumference of the circle is C=2πr which leads to $\frac{C}{r} = 2\pi$, that is the ratio of circumference to radius is a constant. In the same way, the ratio of arclength, s, to r is a constant. We call that constant θ, and define θ to be the radian measure. Thus $\frac{s}{r} = \theta$ where θ, is in radians. Radians is a "dimensionless" unit. Another way that the idea of a radian is typically defined is that 1 radian is the measure of a central angle that subtends an arc whose length equals the length of the radius.



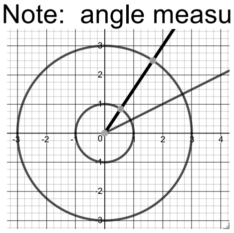
α has radian measure 1



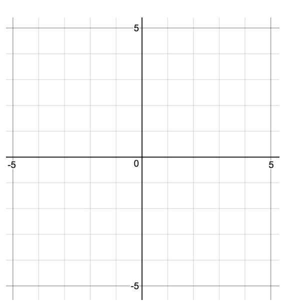
β has radian measure 4

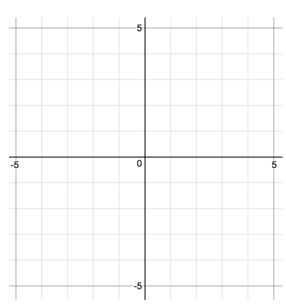
Using this definition, we can determine how many radians are in one revolution. _________________

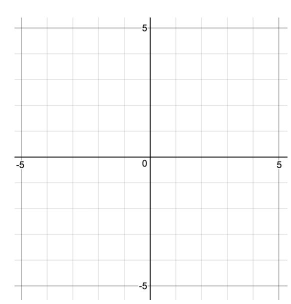
Note: angle measure is not dependent on the size of the circle.



Now using the logic we did with degrees,







Quadrantal Angles ______________ ____________

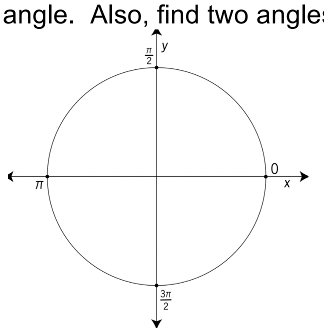
(1/4 of revolution)

(1/8 revolution)
Divide quadrant in half

(1/12 , 2/12 of revolution)
Divide quadrant in thirds

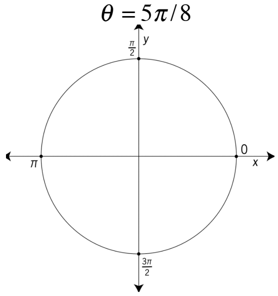
<u>Locating angles in Radians – Same logic but more arithmetic</u>

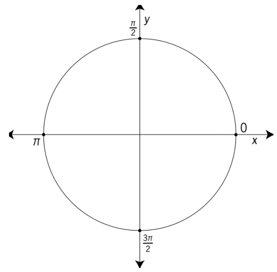
Locate $\theta = 5\pi/6$ by comparing it to quadrantal angles to determine quadrant. Find and use reference angle. Also, find two angles coterminal with $\theta = 5\pi/6$

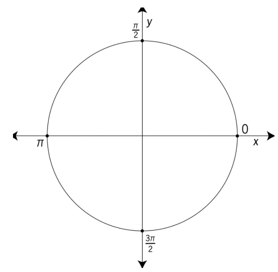


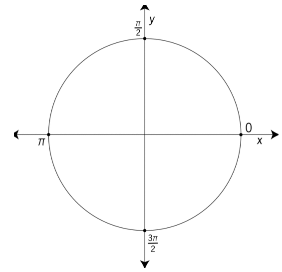
Locate each of the following angles(radians understood unless degrees specified) and find ref. angle.

$\theta = 5\pi/8$ $\theta = 9\pi/7$ $\theta = 21\pi/12$ $\theta = 3$



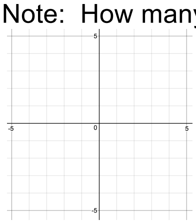






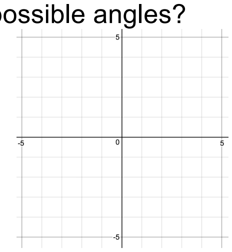
EX: Sketch an angle with reference angle of $\theta = \pi/9$ whose terminal side is in Quadrant 2.

Note: How many possible answers are there? Give an angle coterminal with this angle.



Other times the quadrant is not specified.

EX: Sketch angles with reference angle of $\theta = \pi/3$. How many such terminal sides are there? How many possible angles?



Converting Decimals Degrees: 1 revolution = $2\pi\ radians = 360^\circ$, so conversion factor is $\pi\ radians = 180^\circ$

1) Convert to radians

a) 30° b) 45° c) 60° d) 80°

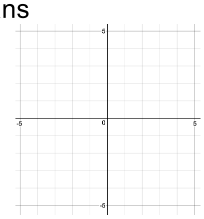
2) Convert to degrees

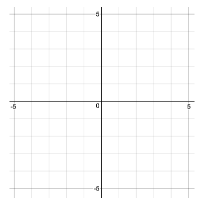
a) $\frac{\pi}{10}$ b) $\frac{5\pi}{12}$ c) 1

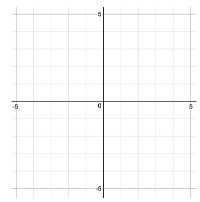
Important Advice: Practice radians so you are comfortable and can "think in radians".

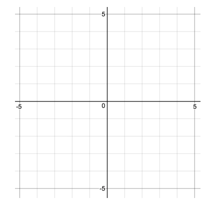
Worksheet: Locating angles and REFERENCE ANGLES - radians

(1) Make a rough sketch of each of the following angles in standard position and give the reference angle in radians

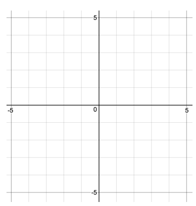

π/10

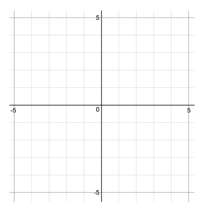

3π/4

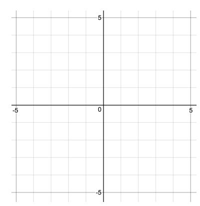

2π/3

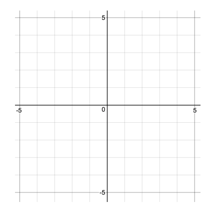

6π/5

Reference Angle:


3π/4


11π/8


23π/12


3

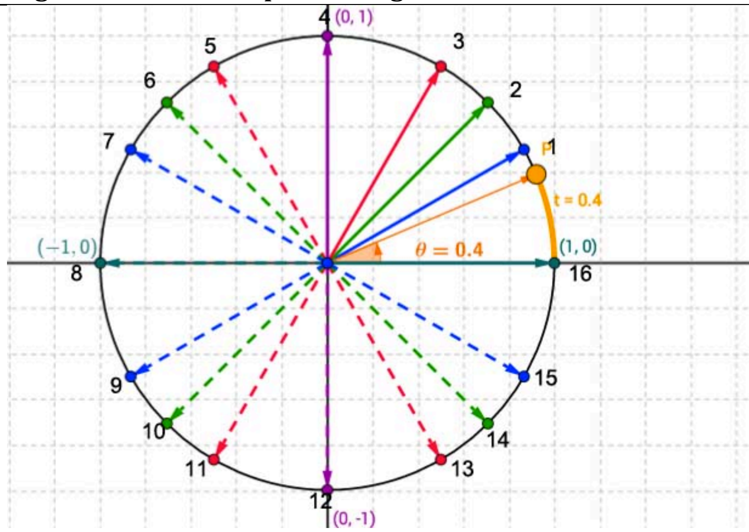
Reference Angle:

---

(2) For each of the following acute angles, find 4 angles, one in each quadrant, having the given angle as a reference angle. Answers should be given in radians

	Q1	Q2	Q3	Q4
π/7				
3π/8				
5π/12				
1				
θ radians				

Worksheet: Getting Familiar with Special Angles - Radians



Given that all the "blue angles" have a reference angle of π/6 radians, write the angle measure for each of the blue angles.

1) ____________________________ (note: the angle numbers are just for reference on this worksheet)

7)____________________________

9) ____________________________

15)____________________________

Given that all the "green angles" all have a reference angle of π/4 radians, write the angle measure in radians for each of the green angles.

2) ____________________________

6)____________________________

10) ____________________________

14)____________________________

Given that all the "red angles" have a reference angle of π/3 radians, write the angle measure in radians for each of the red angles.

3) ____________________________

5)____________________________

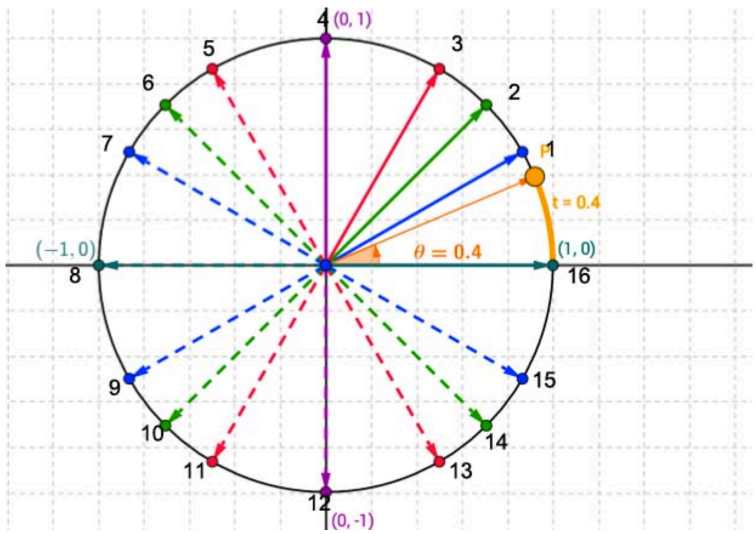
11) ____________________________

13)____________________________

(worksheet cont'd next page)
(worksheet cont'd)

Worksheet: Locating Special Angles Worksheet 2 radians Name: ___________________

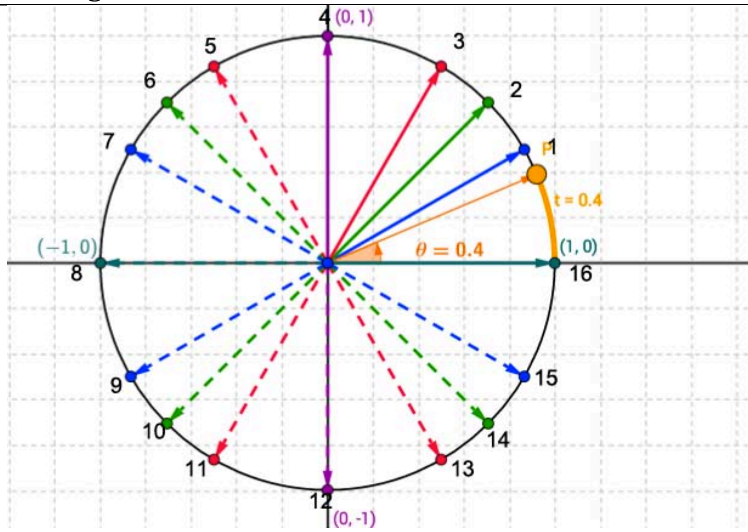
The "blue angles" all have a reference angle of π/6 radians. (see website for colors)
The "green angles" all have a reference angle of π/4 radians. .
The "red angles" all have a reference angle of π/3 radians.
(ignore the orange here)



Locate the following angle and write the corresponding number for each of the following angles. (You need to get quick at this)

π/6 ______________	5π/6 ________________	5π/3 ________________
3π/4 ______________	4π/3 _________________	3π _________________
3π/2 ______________	-7π/6 _________________	-π/2 _______________
−7π/4 ______________	-2π/3 _________________	−13πρ/6 ______________

Worksheet -Special Angles Handout – Mixed



Given that all the "blue angles" have a reference angle of 30 degrees or π/6 radians, write the angle measures, both in radians AND degrees, for each of the blue angles.

1) ____________________ (note: the angle numbers are just for reference on this worksheet)

7)____________________

9) ____________________

15)____________________

Given that all the "green angles" all have a reference angle of 45 degrees or π/4 radians, write the angle measures, both in radians AND degrees, for each of the green angles.

2) ____________________

6)____________________

10) ____________________

14)____________________

Given that all the "red angles" have a reference angle of 60 degrees or π/3 radians, write the angle measures, both in radians AND degrees, for each of the red angles.

3) ____________________

5)____________________

11) ____________________

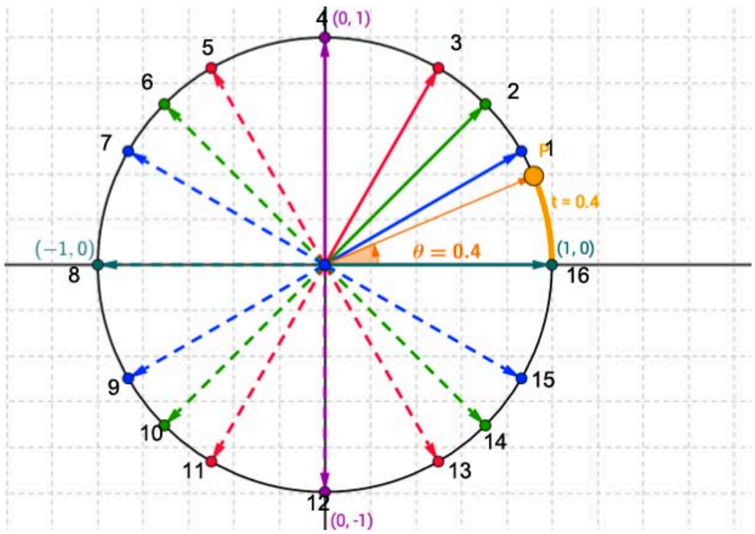
13)____________________

(worksheet cont'd next page)

(worksheet cont'd)

Special Angles Worksheet -Mixed

The "blue angles" all have a reference angle of 30 degrees or π/6 radians.
The "green angles" all have a reference angle of 45 degrees or π/4 radians.
The "red angles" all have a reference angle of 60 degrees or π/3 radians.
(ignore the orange here)

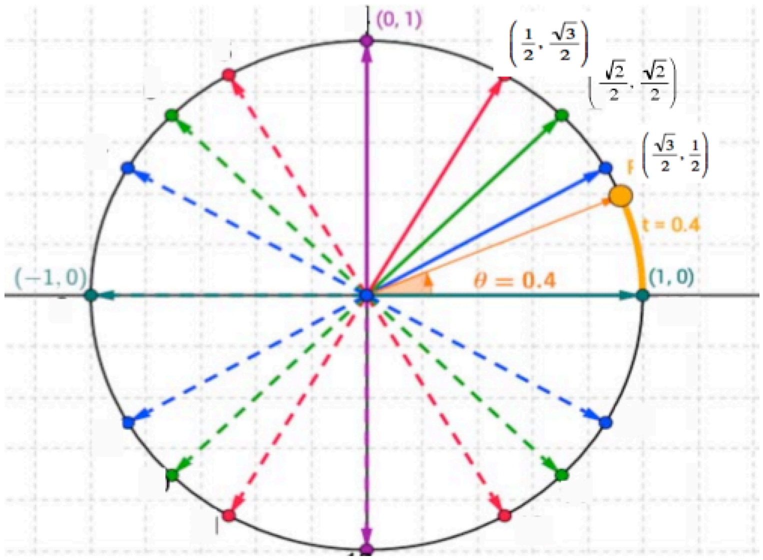


Write the corresponding number for each of the following angles. (You need to get quick at this)

2π/3 ____________	7π/6______________	135°_____________
7π/4 ____________	210°______________	5π/6_____________
5π/3 ____________	-π/4______________	300°_____________
–270° ____________	3π/2 ______________	5π/2_____________
15π/4 ____________	-7π/6______________	225°_____________
-135°____________	4π/3______________	315°_____________
2π/3 ____________	7π/6______________	135°_____________
180° ____________	5π/4 ______________	13π_____________
150° ____________	4π ______________	3π/4_____________

Symmetry and Important Points on the Unit Circle.

We are often interested in looking where the terminal side of some of the “key angles” mentioned earlier intersect the “unit circle” $x^2+y^2=1$ .Notice the symmetry that angles with the same reference angle have (blue->π/6, green->π/4, red->π/3). Suppose the points in the first quadrant were given. Can you fill in the rest?



Alternate helpful graphic. http://www.pccmathuyekawa.com/classes-taught/math_7ab/worksheets/unit%20circle012.pdf *(see math 8 page – unit circle graphic)*

Animation: https://www.desmos.com/calculator/evltrytg3v *(see math 8 page – demos anim. w/triangle)*

Practice knowing all these locations and points

https://www.purposegames.com/game/b58f83e30d *(see math 8 page – unit circle practice)*

Example:

Find the point on the unit circle corresponding to $\theta = \frac{\pi}{3}$ ____________________

Find the point on the unit circle corresponding to $\theta = \frac{5\pi}{6}$ ____________________

Find the point on the unit circle corresponding to $\theta = \frac{3\pi}{2}$ ____________________

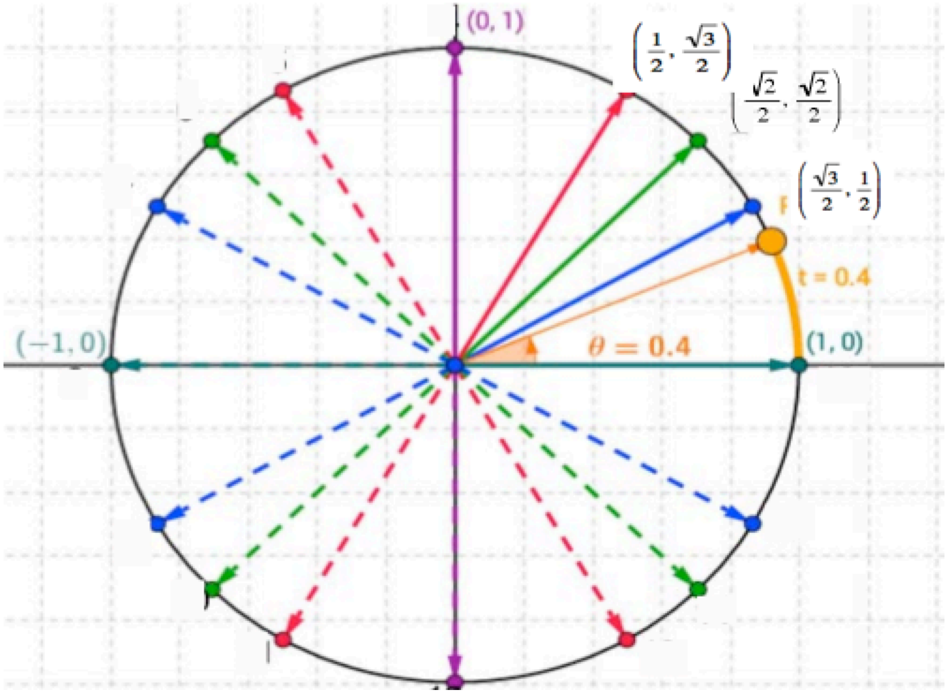
Find the x coordinate of the point on the unit circle corresponding to $\theta = \frac{3\pi}{4}$ ____________________

Find the y coordinate of the point on the unit circle corresponding to $\theta = \frac{-\pi}{6}$ ____________________

How do we find those points in the first quadrant if they are not given?

Worksheet: Finding points on unit circle for key angles.

Find the points on the unit circle corresponding to each of the following angles



The "blue angles" all have a reference angle of 30 degrees or $\pi/6$ radians. The "green angles" all have a reference angle of 45 degrees or $\pi/4$ radians. The "red angles" all have a reference angle of 60 degrees or $\pi/3$ radians.

You will need to be able to do this without the picture above so see what you can do without the picture,

$2\pi/3$ ____________	$7\pi/6$____________	135°____________
$7\pi/4$ ____________	210°____________	$5\pi/6$____________
$5\pi/3$ ____________	$-\pi/4$____________	300°____________
–270° ____________	$3\pi/2$ ____________	$5\pi/2$____________
$15\pi/4$ ____________	$-7\pi/6$____________	225°____________
-135°____________	$4\pi/3$____________	315°____________
$2\pi/3$ ____________	$7\pi/6$____________	135°____________
180° ____________	$5\pi/4$ ____________	13π____________
150° ____________	4π ____________	$3\pi/4$____________

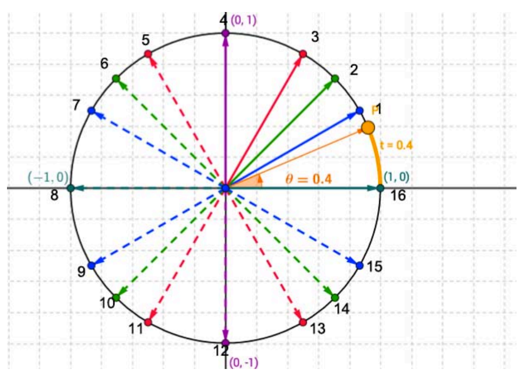
SAMPLE TEST

NOTE: Sample tests are not meant to be a complete study guide. This is just a test that was given on this material one time. Yours will be similar in length and difficulty but will not be exactly the same. There may be topics from the homework that are not covered on this test but WILL be on your test. Working these problems, without referring to notes or solutions should be only ONE PART of your study.

- **Notebook should be turned in before test. It will not be accepted after.**
- **Phones must be turned OFF and put away. Any visible phone (smart watch, headphones, ipad etc.) will result in a grade F .**
- **No scratch paper or notes.**
- **No graphing calculator.**
- **No credit will be given for solutions if work is not shown.**
- **I expect clear and legible presentations .**

(1) Same figure as on homework, see board for colors.

The "blue angles" all have a reference angle of 30 degrees or π/6 radians.
The "green angles" all have a reference angle of 45 degrees or π/4 radians.
The "red angles" all have a reference angle of 60 degrees or π/3 radians.
(ignore the orange here) (12 points)



Write the corresponding number for each of the following angles:

150°______________ 7π/6________________ 210°_______________

7π/4 ______________ 330°______________ 11π/6______________

4π/3 ______________ 3π/2________________ -π/3_______________

-330°______________ 2π/3 ______________ 5π ______________

What are the coordinates of the points at: 3 points

1) 7)

2) 11)

3) 14)

(2) Solve using any of the methods discussed in class. (10 points)

$$\begin{cases} 2x - y + z = 4 \\ x + 3y + 2z = -1 \\ 7x \quad + 5z = 11 \end{cases}$$

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(3) Use <u>Cramer's Rule</u> to solve the following system. $\begin{cases} x \ + 3y + z = 2 \\ x + y + 2z = 1 \\ 2x + 3y + 4z = 3 \end{cases}$ (8 points)

(No credit given for a different method)

(4) Given the following matrices: (a-d, 2 points each; e,f 4 points each)

$$A=\begin{bmatrix} 8 & 3 \\ -1 & -2 \end{bmatrix} \quad B=\begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & -3 & 2 \\ 0 & 5 & 1 & -1 \\ 0 & 2 & 1 & 1 \end{bmatrix} \quad C=\begin{bmatrix} 9 & 3 \\ -4 & 1 \end{bmatrix} \quad D=\begin{bmatrix} -1 & 7 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

Find the following, if possible. (If not possible, say so.)

(a) A + C

(b) AC

(c) BC

(d) det(C)

(e) AD

(f) det (B)

(5) Given $A = \begin{bmatrix} 0 & -1 & \frac{1}{2} \\ 3 & -4 & 3 \\ 1 & -2 & 3 \end{bmatrix}$

(a) Find A^{-1} (10 points)

(b) Solve the system of equations by writing it as a matrix equation Ax=B and using the inverse of the coefficient matrix (which you found in part a).

$$\begin{cases} -y + \frac{1}{2}z = 7 \\ 3x - 4y + 3z = 1 \\ x - 2y + 3z \quad = 2 \end{cases}$$

(3 points)

(6) (a) Convert from DMS (degree, minute seconds)to decimal degrees, show work. $19^\circ 45' 72''$
(8 points)

(b) Convert from decimal degrees to DMS , show work. 42.6°

(c) Convert from radians to degrees: $\frac{7\pi}{9}$

(d) Convert from degrees to radians, exactly (no calculator): 12°

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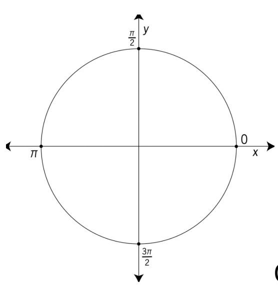
(7) Graph the angle $\theta = 7\pi/12$ in standard position. Give two coterminal angles, one of which is positive and the other negative. Find the reference angle. (8 points)



Coterminal positve _______ Coterminal negative ________ Ref angle_____

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(8) (For each of the following acute angles, find 4 angles, one in each quadrant, having the given angle as a reference angle. Answer in the units given, exactly. (12 points)

	Q1	Q2	Q3	Q4
23°				
$2\pi/5$				
0.2				

(7) Use matrix methods (Gaussian elimination or Gauss Jordan) to solve: (10 points)

$$\begin{cases} -x-2y-z=-3 \\ 2x+y+z=16 \\ x+y+2z=9 \end{cases}$$

You must obtain row echelon form or reduced row echelon form. Be sure to label operations performed at each step.